Eur. Phys. J. C 22, 601–611 (2001)
Digital Object Identifier (DOI) 10.1007/s100520100807

THE EUROPEAN
PHYSICAL JOURNAL C

# Quantum mechanics on Riemannian manifold in Schwinger's quantization approach IV

## Quantum mechanics of superparticle

N.M. Chepilko[1,a], A.V. Romanenko[2,b]

[1] Physics Institute of the Ukrainian Academy of Sciences, Kyiv-03 028, Ukraine
[2] Kyiv Taras Shevchenko University, Department of Physics, Kyiv-03 022, Ukraine

Received: 22 June 2001 / Revised version: 4 October 2001 /
Published online: 23 November 2001 – 

**Abstract.** In this paper we extend Schwinger's quantization approach to the case of a supermanifold considered as a coset space of the Poincaré group by the Lorentz group. In terms of coordinates parameterizing a supermanifold, quantum mechanics for a superparticle is constructed. As in models related to the usual Riemannian manifold, the key role in the analysis is played by Killing vectors. The main feature of quantum theory on the supermanifold consists of the fact that the spatial coordinates do not commute and therefore are represented on wave functions by integral operators.

## 1 Introduction

The present paper completes our series of works (see [1–3]) devoted to extending Schwinger's quantization procedure to the case of quantum theory defined on manifolds with a group structure. The aim of the present paper is to develop the quantum mechanics for a particle moving on a superspace treated as a coset space

$$^s\mathbb{M}^n \simeq {}^s\mathsf{ISO}(p,q)/\mathsf{SO}(p,q), \quad \text{where } p+q=n,$$

where $\mathsf{SO}(p,q)$ denotes the special orthogonal rotation group and $^s\mathsf{ISO}(p,q)$ is a supersymmetrical extension of the Poincaré group $\mathsf{ISO}(p,q)$. A local coordinate system in a superspace consists of the variables $z^A = (x^\mu, \psi, \overline{\psi})$, where $x^\mu$, $\mu = \overline{1,n}$ denote spatial coordinates of the underlying pseudo-Euclidean space $\mathbb{M}^n \simeq \mathsf{ISO}(p,q)/\mathsf{SO}(p,q)$, and $\psi$, $\overline{\psi}$ are Dirac bispinors, whose inner structure is in accordance with the dimension and the signature of the underlying manifold $\mathbb{M}^n$. We briefly examine the main features of the definition of a superspace in Sect. 2.

The quantum mechanics of a superparticle is constructed in terms of a logical scheme presented in [1–3]. To obtain the explicit form of the permissible variations of the coordinates we have analyzed the symmetries of the Lagrangian which determines the dynamical behavior of a superparticle. The results obtained at the classical level are briefly discussed in Sects. 3 and 4. As appeared in [1–3], the permissible variations $\delta z^A$ are closely related to Killing vector fields defined, however, on $^s\mathbb{M}^n$ instead of $\mathbb{M}^n$.

[a] e-mail: chepilko@zeos.net
[b] e-mail: ar@ups.kiev.ua

In Sect. 5 the generator of the permissible variations is constructed. Using it we derive the commutation relations for the quantum-mechanical operators that are presented in Sect. 6. The main feature of the algebra of the commutation relations is that the coordinates $z^A$ (including the spatial ones) do not commute. The fact that $[x^\mu, x^\nu] \neq 0$ means that the space-time coordinates $x^\mu$ in quantum theory lose the meaning which they have in $\mathbb{M}^4$. The presence of odd variables $\psi$ and $\overline{\psi}$ modifies the geometric nature of the space-time coordinates. In Sect. 6 we also discuss the question of constructing the total set of commuting variables which determines the way a description can be given of the physical space of states.

Using the total set given in Sect. 6, we develop the coordinate representation in Sect. 7. Namely, we obtain the representation of operators on the space of wave functions and the form of the Schrödinger equation. The essential feature of the coordinate representation consists of the fact that the spatial coordinates $x^\mu$ are represented by integral operators. Possibly, this result is a hint to the existence of some effective spatial size of a superparticle. This fact should explain the decrease of divergences in the integrals of motion in supersymmetrical quantum field theories.

## 2 Supersymmetry and superspace

Supersymmetry realizes the formal connection between bosonic and fermionic states of quantum systems. The basic principles of supersymmetry have been discovered in 1970s in a series of works [4,5].

The aim of the present section is to examine the definition of a superspace and explain some notation used in the text. As in the case of other symmetries, supersymmetry can be formulated in terms of a transformation group acting on physical states. But taking into account the features of Fermi statistics, one can observe that the framework of classical Lie groups and algebras becomes inconsistent in the description of a supersymmetry. Such a problem can be solved with the use of $\mathbb{Z}_2$-graded Lie algebras (see [7]).

$\mathbb{Z}_2$-grading of the algebra $\mathcal{A}$ means the existence of a map

$$\mathfrak{a} : \mathcal{A} \longrightarrow \mathbb{Z}_2,$$

where $\mathbb{Z}_2 \equiv \{0, 1\}$. The algebra $\mathcal{A}$ can be divided into two parts: the *even* elements $\mathcal{A}_{\mathrm{e}} = \{a \in \mathcal{A} : \mathfrak{a}(a) = 0\}$ and the *odd* elements $\mathcal{A}_{\mathrm{o}} = \{a \in \mathcal{A} : \mathfrak{a}(a) = 1\}$, so that $\mathcal{A} = \mathcal{A}_{\mathrm{e}} \oplus \mathcal{A}_{\mathrm{o}}$.

The definition of a Lie bracket can be modified for the case of graded algebras as follows:

$$[a, b] := ab - s(a, b)ba, \quad \forall a, b \in \mathcal{A}, \tag{1}$$

where $s(a, b)$ denotes the signature factor

$$s(a, b) = (-1)^{\mathfrak{a}(a)\cdot\mathfrak{a}(b)} = \begin{cases} -1, \text{ when } a,\ b \text{ are odd,} \\ +1, \text{ in all the other cases.} \end{cases}$$

The generalized Lie bracket is called the *supercommutator* and satisfies the following identities

$$\begin{aligned} [a, b] &= -s(a, b)[b, a], \\ [a, b_1 + b_2] &= [a, b_1] + [a, b_2], \\ [a, bc] &= [a, b]c + s(a, b)b[a, c], \\ s(a, c)[a, [b, c]] + s(b, a)[b, [c, a]] &+ s(c, b)[c, [a, b]] = 0. \end{aligned} \tag{2}$$

In a flat space the physical features of a theory are determined by the transformation properties of the objects under the actions of the group of symmetries of the pseudo-Euclidean space $\mathbb{M}^n$ equipped with the diagonal metric

$$\{\eta_{\mu\nu}\} = \mathrm{diag}(\underbrace{1, \ldots, 1}_{p}, \underbrace{-1, \ldots, -1}_{q}), \quad \mu, \nu = \overline{1, n}.$$

This group is the $n(n + 1)/2$-parameter inhomogeneous Lie group $\mathsf{ISO}(p, q)$. In the case $n = 4$, $p = 1$, $q = 3$ and

$$\eta = \mathrm{diag}(1, -1, -1, -1)$$

it coincides with $\mathsf{ISO}(1, 3) = \mathsf{SO}(1, 3) \ltimes \mathsf{T}_4$, and the corresponding Lie algebra satisfies the following commutation relations:

$$[P_\mu, P_\nu] = 0, \tag{3a}$$

$$[J_{\mu\nu}, P_\alpha] = \mathrm{i}(\eta_{\mu\alpha}P_\nu - \eta_{\nu\alpha}P_\mu), \tag{3b}$$

$$[J_{\mu\nu}, J_{\alpha\beta}] = \mathrm{i}\left(\eta_{\mu\alpha}J_{\nu\beta} + \eta_{\nu\beta}J_{\mu\alpha} - \eta_{\nu\alpha}J_{\mu\beta} - \eta_{\mu\beta}J_{\nu\alpha}\right). \tag{3c}$$

Here $P_\mu$ and $J_{\mu\nu}$ denote the generators of translations and $\mathsf{SO}(1, 3)$-rotations respectively. Due to the structure of the semidirect product of the Poincaré group, the space $\mathbb{M}^4$ is isomorphic to $\mathsf{T}_4$ and therefore can be defined as a quotient space

$$\mathbb{M}^4 \simeq \mathsf{ISO}(1, 3)/\mathsf{SO}(1, 3). \tag{4}$$

Denote the coordinates of $\mathbb{M}^4$ by $\{x^\mu\}$. Then the transformations generated by

$$\varepsilon^\mu P_\mu + \frac{1}{2}\varepsilon^{\mu\nu}J_{\mu\nu} \in \mathfrak{iso}(1, 3),$$

with $\varepsilon^{\mu\nu} = -\varepsilon^{\nu\mu} = \mathrm{const}$, $\varepsilon^\mu = \mathrm{const}$, causes the following changes of space-time coordinates:

$$x^\mu \to x'^\mu = x^\mu + \varepsilon^\mu + \varepsilon^\mu{}_\nu x^\nu. \tag{5}$$

The main idea on which the supersymmetry is based consists of the usage in (4) of the extension of the group $\mathsf{ISO}(1, 3)$ instead of $\mathsf{ISO}(1, 3)$. Such an extension can be developed by adding to the Lie algebra new generators that must correspond to some representation of $\mathsf{SO}(1, 3)$. The simplest way of doing this is to include spinor generators $Q$, $\overline{Q}$ with anticommutative components with the following properties:

$$[P_\mu, Q_a] = 0, \quad [P_\mu, \overline{Q}^a] = 0, \tag{6a}$$

$$[Q_a, J_{\mu\nu}] = -(\sigma_{\mu\nu})^a{}_b Q^b, \quad [\overline{Q}^a, J_{\mu\nu}] = \overline{Q}^b(\sigma_{\mu\nu})^a{}_b, \tag{6b}$$

$$[Q_a, \overline{Q}^b] = (\gamma^\mu)_a{}^b P_\mu. \tag{6c}$$

where $\sigma_{\mu\nu} = (\mathrm{i}/4)[\gamma_\mu, \gamma_\nu]$. The algebra in (3) and (6) coincides with the $N = 1$ supersymmetrical extension of the Poincaré algebra. The coset space

$${}^s\mathbb{M}^4 \simeq {}^s\mathsf{ISO}(1, 3)/\mathsf{SO}(1, 3)$$

is called a *superspace* (here ${}^s\mathsf{ISO}(1, 3)$ denotes the corresponding extended group). The coordinates of ${}^s\mathbb{M}^4$ are denoted $(x^\mu, \psi, \overline{\psi})$, where $\psi$ and $\overline{\psi} \equiv \psi^\dagger\gamma^o$ are Dirac bispinors with anticommutative components, and $x^\mu$ are the coordinates of the space $\mathbb{M}^4$; this is called an underlying space for ${}^s\mathbb{M}^4$.

After introducing the unified notation $z^A = (x^\mu, \psi, \overline{\psi})$ and taking into account the anticommutativity of $\psi$ and $\overline{\psi}$ we can write

$$z^A \cdot z^B = s(z^A, z^B)z^B \cdot z^A. \tag{7}$$

The transformations induced by the generator $G = \overline{\xi}Q + \overline{Q}\xi$, where $\xi$, $\overline{\xi}$ are Grassmannian spinor constants, have the form

$$x^\mu \longrightarrow x'^\mu = x^\mu + \frac{\mathrm{i}}{2}\left(\overline{\psi}\gamma^\mu\xi - \overline{\xi}\gamma^\mu\psi\right), \tag{8a}$$

$$\psi \longrightarrow \psi' = \psi + \xi, \quad \overline{\psi} \longrightarrow \overline{\psi}' = \overline{\psi} + \overline{\xi}. \tag{8b}$$

These transformations are called *supertranslations*. One can observe from (6) that supertranslations commute with the usual translations generated by $\varepsilon^\mu P_\mu$.

In the superspace with the coordinates $z^A = (x^\mu, \psi, \overline{\psi})$ the 1-forms $\mathrm{d}x^\mu$ are not invariant under the supertranslations (8) and therefore cannot be chosen as a basis of tensor fields. Let us consider the transformation laws of $\mathrm{d}z^A$ in the case of supertranslations with the parameters $\xi$ and $\overline{\xi}$. Since

$$\mathrm{d}z^A \longrightarrow \mathrm{d}z'^A = \frac{\partial z'^A}{\partial z^B}\mathrm{d}z^B,$$

then

$$\mathrm{d}\psi \longrightarrow \mathrm{d}\psi, \quad \mathrm{d}\overline{\psi} \longrightarrow \mathrm{d}\overline{\psi},$$

$$\mathrm{d}x^\mu \longrightarrow \mathrm{d}x^\mu + \frac{\mathrm{i}}{2}\left(\mathrm{d}\overline{\psi}\gamma^\mu\xi - \overline{\xi}\gamma^\mu\mathrm{d}\psi\right).$$

Using the trivial equalities $\delta\psi = \psi' - \psi = \xi$, $\delta\psi = \overline{\psi}\,' - \overline{\psi} = \overline{\xi}$ we arrive at the conclusion that 1-forms invariant under supertranslations (8) coincide with

$$e^\mu := \mathrm{d}x^\mu + \frac{\mathrm{i}}{2}\left(\mathrm{d}\overline{\psi}\gamma^\mu\psi - \overline{\psi}\gamma^\mu\mathrm{d}\psi\right). \tag{9}$$

Together with $\mathrm{d}\psi$ and $\mathrm{d}\overline{\psi}$ they form the basis of invariant 1-forms on ${}^s\mathbb{M}^4$.

A similar analysis can be carried out for the vector fields $\partial/\partial z^A$. The transformation law reads

$$\frac{\partial}{\partial z^A} = \frac{\partial z'^B}{\partial z^A}\frac{\partial}{\partial z'^B}.$$

Hence

$$\frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial x'^\mu}, \quad \frac{\partial}{\partial \psi} = \frac{\partial}{\partial \psi'} - \frac{\mathrm{i}}{2}\overline{\xi}\gamma^\mu\frac{\partial}{\partial x'^\mu},$$
$$\frac{\partial}{\partial \overline{\psi}} = \frac{\partial}{\partial \overline{\psi}\,'} + \frac{\mathrm{i}}{2}\gamma^\mu\xi\frac{\partial}{\partial x'^\mu}.$$

Using $\delta\psi = \psi' - \psi = \xi$, $\delta\overline{\psi} = \overline{\psi}\,' - \overline{\psi} = \overline{\xi}$, we find the following invariant vector fields:

$$\overline{D} = \frac{\partial}{\partial\psi} - \frac{\mathrm{i}}{2}\overline{\psi}\gamma^\mu\frac{\partial}{\partial x^\mu}, \quad D = \frac{\partial}{\partial\overline{\psi}} + \frac{\mathrm{i}}{2}\gamma^\mu\psi\frac{\partial}{\partial x^\mu}. \tag{10}$$

Together with $\partial/\partial x^\mu$ they form the basis of invariant vector fields on ${}^s\mathbb{M}^4$.

The expressions for the invariant 1-forms can be represented as

$$e^A = E^A_B\mathrm{d}z^B, \quad e_A = \overline{E}^B_A\frac{\partial}{\partial z^B}, \tag{11}$$

where

$$\{E^A_B\} = \begin{pmatrix} \delta^\mu_\nu & -\frac{\mathrm{i}}{2}(\overline{\psi}\gamma^\mu)^a & -\frac{\mathrm{i}}{2}(\gamma^\mu\psi)_a \\ 0 & \delta^a_b & 0 \\ 0 & 0 & \delta^a_b \end{pmatrix},$$

$$\{\overline{E}^A_B\} = \begin{pmatrix} \delta^\mu_\nu & 0 & 0 \\ -\frac{\mathrm{i}}{2}(\overline{\psi}\gamma^\mu)^a & \delta^a_b & 0 \\ \frac{\mathrm{i}}{2}(\gamma^\mu\psi)_a & 0 & \delta^a_b \end{pmatrix}. \tag{12}$$

Here we assume that the differentials $\mathrm{d}\psi$, $\mathrm{d}\overline{\psi}$ are right multipliers in (11).

These constructions can easily be generalized for the $n > 4$ dimensional case. The superspace ${}^s\mathbb{M}^n$ with the coordinates $(x^\mu, \psi, \overline{\psi})$ is described by the coordinates of the underlying manifold ${}^s\mathbb{M}^n$ and $2^{[n/2]}$-component Dirac bispinors with the Grassmannian components. The Clifford algebra is generated by the relation for the basic elements

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\nu = 2\eta^{\mu\nu}, \tag{13}$$

where $\eta_{\mu\nu}$ is the metric of $\mathbb{M}^n$.

The supertranslations have the form

$$x^\mu \longrightarrow x'^\mu + \frac{\mathrm{i}}{2}\left(\overline{\psi}\gamma^\mu\xi - \overline{\xi}\gamma^\mu\psi\right), \tag{14}$$

$$\psi \longrightarrow \psi' = \psi + \xi, \quad \overline{\psi} \longrightarrow \overline{\psi}\,' = \overline{\psi} + \overline{\xi}. \tag{15}$$

In the $n$-dimensional case the invariant 1-forms and vector fields coincide with (9) and (10).

## 3 Lagrangian for point particle in superspace

As an initial step we choose the Maurer–Cartan 1-form (9). The velocity of a superparticle can be defined in terms of this form by

$$e^\mu = v^\mu\mathrm{d}\tau,$$

where $\tau$ denotes the evolution parameter (for example, the physical time). After introducing the symbol "$\partial$" that denotes the derivative with respect to $\tau$, the velocity can be rewritten as

$$v^\mu = \partial x^\mu + \frac{\mathrm{i}}{2}\left(\overline{\psi}\gamma^\mu\partial\psi - \partial\overline{\psi}\gamma^\mu\psi\right), \tag{16}$$

where the coordinates $(x^\mu, \psi, \overline{\psi})$ describing the trajectory of a superparticle in a superspace are functions of $\tau$. Treating (16) as a vector on the underlying manifold $\mathbb{M}^n$ let us define the Lagrangian by

$$L = \frac{1}{2}\eta_{\mu\nu}v^\mu v^\nu. \tag{17}$$

The equations of motion can be obtained from the stationary action principle by taking the variation of the functional

$$S[x, \psi, \overline{\psi}] = \int_{\tau_1}^{\tau_2} L(x, \psi, \overline{\psi})\mathrm{d}\tau$$

with respect to the variables $z^A = (x^\mu, \psi, \overline{\psi})$ under the condition

$$\delta z^A(\tau)\big|_{\tau=\tau_{1,2}} = 0.$$

Consider the explicit variation of $L$ extracting a total time derivative. Evidently, for the synchronous variations

$\delta z^A$ (i.e. $\partial\delta z^A = \delta\partial z^A$) it can be written as

$$\begin{aligned}\delta L = \eta_{\mu\nu} v^\mu \delta v^\nu &= \partial\left[p_\mu\left(\delta x^\mu + \frac{\mathrm{i}}{2}\left(\overline{\psi}\gamma^\mu\delta\psi - \delta\overline{\psi}\gamma^\mu\psi\right)\right)\right] \\ &- \partial p_\mu \delta x^\mu + \frac{\mathrm{i}}{2}\delta\overline{\psi}\left[p_\mu\gamma^\mu\partial\psi + \partial(p_\mu\gamma^\mu\psi)\right] \\ &- \frac{\mathrm{i}}{2}\left[\overline{\psi}\gamma^\mu p_\mu + \partial(\overline{\psi}\gamma^\mu p_\mu)\right]\delta\psi \\ &\equiv \partial\left[p_\mu\left(\delta x^\mu + \frac{\mathrm{i}}{2}\left(\overline{\psi}\gamma^\mu\delta\psi - \delta\overline{\psi}\gamma^\mu\psi\right)\right)\right] - \delta x^\mu\partial p_\mu \\ &+ \frac{\mathrm{i}}{2}\left[2\hat{p}\partial\psi + (\partial\hat{p})\psi\right] - \frac{\mathrm{i}}{2}\left[2\partial\overline{\psi}\hat{p} + \overline{\psi}\partial\hat{p}\right], \end{aligned} \tag{18}$$

where we denote

$$p_\mu = \eta_{\mu\nu} v^\nu, \quad \hat{p} = p_\mu \gamma^\mu. \tag{19}$$

Taking into account the arbitrariness of the variations $\delta z^A$ one can obtain the equations of motion in the following form:

$$\partial p_\mu = 0, \quad \hat{p}\partial\psi + \frac{1}{2}(\partial\hat{p})\psi = 0, \quad (\partial\overline{\psi})\hat{p} + \frac{1}{2}\overline{\psi}\partial\hat{p} = 0. \tag{20}$$

Making a substitution of the first equation into the other ones we get

$$\partial p_\mu = 0, \quad \hat{p}\partial\psi = 0, \quad \partial\overline{\psi}\hat{p} = 0. \tag{21}$$

The same result can be obtained from the Euler–Lagrange equations which have to be used with care because of the presence of Grassmann variables. A little complication arises due to the two different ways of giving the definition of the derivation with respect to $\psi$ and $\overline{\psi}$ (left and right derivatives). These definitions appear when the factor $\delta z$ is put outside the brackets in the variation of an arbitrary function $F(z)$ in two different ways, namely

$$\delta F(z) = F(z+\delta z) - F(z) = \delta z^A \frac{\boldsymbol{\partial} F}{\boldsymbol{\partial} z^A} = \frac{\overleftarrow{\partial} F}{\overleftarrow{\partial} z^A}\delta z^A; \tag{22}$$

here the symbols $\boldsymbol{\partial}$ and $\overleftarrow{\partial}$ denote the left and right derivation respectively, and

$$\begin{aligned}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} z}\left(f_1(z)f_2(z)\right) &= \frac{\boldsymbol{\partial} f_1(z)}{\boldsymbol{\partial} z} f_2(z) + (-1)^{\mathfrak{a}(f_1)} f_1 \frac{\boldsymbol{\partial} f_2}{\boldsymbol{\partial} z}, \\ \frac{\overleftarrow{\partial}}{\overleftarrow{\partial} z}\left(f_1(z)f_2(z)\right) &= (-1)^{\mathfrak{a}(f_2)}\frac{\overleftarrow{\partial} f_1(z)}{\overleftarrow{\partial} z} f_2(z) \\ &+ f_1\frac{\overleftarrow{\partial} f_2}{\overleftarrow{\partial} z}.\end{aligned}$$

It is easy to prove that

$$\frac{\boldsymbol{\partial} F}{\boldsymbol{\partial} z} = -(-1)^{\mathfrak{a}(F)}\frac{\overleftarrow{\partial} F}{\overleftarrow{\partial} z}.$$

The derivatives with respect to odd variables are anticommutative [7]. We assume that the derivative with respect to $\psi$ and $\overline{\psi}$ from now on mean the right- and left-side derivation, respectively.

The generalized momenta conjugate to $x^\mu$, $\psi$, $\overline{\psi}$ are

$$\mathcal{P}_\mu = \frac{\partial L}{\partial\partial x^\mu} = \eta_{\mu\nu} v^\mu \equiv p_\mu, \tag{23a}$$

$$\mathcal{P}_\psi = \frac{\partial L}{\partial\partial\psi} = \frac{\mathrm{i}}{2}\overline{\psi}\hat{p}, \tag{23b}$$

$$\mathcal{P}_{\overline{\psi}} = \frac{\partial L}{\partial\partial\overline{\psi}} = -\frac{\mathrm{i}}{2}\hat{p}\psi. \tag{23c}$$

A direct calculation shows that the Hamiltonian coincides with the Lagrangian. This fact means that the theory under consideration is purely kinematic.

The Lagrangian after some algebraic calculations can be rewritten in the following form:

$$L = \frac{1}{2}\partial z^A g_{AB}(z)\partial z^B; \tag{24}$$

the structure of the "metric" $g_{AB}$ can be obtained from (17) after the direct substitution of (16):

$$\{g_{AB}\} = \begin{pmatrix} \eta_{\mu\nu} & \frac{\mathrm{i}}{2}(\overline{\psi}\gamma_\mu)_a & \frac{\mathrm{i}}{2}(\gamma_\mu\psi)^a \\ \frac{\mathrm{i}}{2}(\overline{\psi}\gamma_\mu)_a & \frac{1}{4}(\overline{\psi}\gamma^\mu)_a\eta_{\mu\nu}(\overline{\psi}\gamma^\nu)_b & \frac{1}{4}(\gamma^\mu\psi)^a\eta_{\mu\nu}(\overline{\psi}\gamma^\nu)_b \\ -\frac{\mathrm{i}}{4}(\gamma_\mu\psi)^a & \frac{1}{4}(\overline{\psi}\gamma^\mu)_a\eta_{\mu\nu}(\gamma^\nu\psi)^b & \frac{1}{4}(\gamma^\mu\psi)^a\eta_{\mu\nu}(\gamma^\nu\psi)^b \end{pmatrix}.$$

One can arrive at the same result after two-fold differentiation of the Lagrangian with respect to the velocities taking into account the side of the derivation.

Note that there is the following equivalent form of the Lagrangian that can be obtained with the use of the definitions (11) and (12):

$$\begin{aligned}L &= \frac{1}{2}v^\mu\eta_{\mu\nu}v^\nu = \frac{1}{2}(E^\mu_A\partial z^A)\eta_{\mu\nu}(E^\nu_B\partial z^B) \\ &= \frac{1}{2}s(E^\mu_A, z^A)\partial z^A\left(E^\mu_A\eta_{\mu\nu}E^\nu_B\right)\partial z^B.\end{aligned}$$

## 4 Classical symmetries of Lagrangian for superparticle

The infinitesimal coordinate transformation $z^A \to z'^A = z^A + \delta z^A$ in the superspace is a symmetry of the dynamical theory when $\delta L$ is equal to zero or can be represented as a total time derivative of some function. In the case of mechanical systems it is possible to redefine the Lagrangian maintaining the character of a theory in order to achieve $\delta L = 0$. We assume that such a property of the Lagrangian holds.

Taking into account the definition of the Lagrangian we can write in the case of synchronous variations ($\partial\delta z = \delta\partial z$)

$$\begin{aligned}\delta L = \frac{1}{2}&\left[\partial\delta z^A g_{AB}\partial z^B + \partial z^A g_{AB}\partial\delta z^B\right. \\ &\left.+\ \partial z^A\left(\frac{\partial g_{AB}}{\partial z^C}\delta z^C\right)\partial z^B\right].\end{aligned}$$

Since $\delta z^A = \delta z^A(z)$, then $\delta\partial z^A = \partial\delta z^A = \delta z^C \partial\delta z^A/\partial z^C$, so that

$$\delta L = \partial z^A \left[g_{CB}\partial_A \delta z^C + g_{AC}\partial_B \delta z^C + \delta z^C \partial_C g_{AB}\right]\partial z^B = 0. \tag{25}$$

Hence the symmetries of the Lagrangian are isometries of the metric $\{g_{AB}\}$ and the variations $\delta z^A$ obey the Killing equations.

In order to write down the explicit form of the Killing equations it is suitable to calculate directly the variation of $L$ under the transformation $z \to z' = z + \delta z(z)$. Using (16) we find

$$\delta L = v^\mu \eta_{\mu\nu} \delta v^\nu \equiv p_\mu \delta v^\mu. \tag{26}$$

It is not difficult to express $\delta v^\mu$ in terms of the variations $\delta x^\mu(x, \psi, \overline{\psi})$, $\delta\psi(x, \psi, \overline{\psi})$ and $\delta\overline{\psi}(x, \psi, \overline{\psi})$, taking into account the properties of derivation operations:

$$\begin{aligned}\delta v^\mu &= \partial x^\nu \left[\frac{\partial \delta x^\mu}{\partial x^\nu} + \frac{\mathrm{i}}{2}\left(\overline{\psi}\gamma^\mu \frac{\partial\delta\psi}{\partial x^\nu} - \frac{\partial\delta\overline{\psi}}{\partial x^\nu}\gamma^\mu\psi\right)\right] \\ &+ \partial\overline{\psi}\left[\frac{\partial\delta x^\mu}{\partial\overline{\psi}} - \frac{\mathrm{i}}{2}\left(\gamma^\mu\delta\psi + \overline{\psi}\gamma^\mu\frac{\partial\delta\psi}{\partial\overline{\psi}} + \frac{\partial\delta\overline{\psi}}{\partial\overline{\psi}}\gamma^\mu\psi\right)\right] \\ &+ \left[\frac{\partial\delta x^\mu}{\partial\psi} + \frac{\mathrm{i}}{2}\left(\delta\overline{\psi}\gamma^\mu + \overline{\psi}\gamma^\mu\frac{\partial\delta\psi}{\partial\psi} + \frac{\partial\delta\overline{\psi}}{\partial\psi}\gamma^\mu\psi\right)\right]\partial\psi.\end{aligned} \tag{27}$$

Let us rewrite (26) using (27) as follows:

$$\delta L = v^\mu \left(A_{\mu\nu}\partial x^\nu + \partial\overline{\psi}B_\mu + \overline{B}_\mu \partial\psi\right), \tag{28}$$

where the coefficients multiplying the velocities $\partial x^\mu$, $\partial\psi$ and $\partial\overline{\psi}$ are denoted

$$A_{\mu\nu} = \frac{\partial\delta x_\mu}{\partial x^\nu} + \frac{\mathrm{i}}{2}\left(\overline{\psi}\gamma_\mu\frac{\partial\delta\psi}{\partial x^\nu} - \frac{\partial\delta\overline{\psi}}{\partial x^\nu}\gamma_\mu\psi\right), \tag{29a}$$

$$B_\mu = \frac{\partial\delta x_\mu}{\partial\overline{\psi}} - \frac{\mathrm{i}}{2}\left(\gamma_\mu\delta\psi + \overline{\psi}\gamma_\mu\frac{\partial\delta\psi}{\partial\overline{\psi}} + \frac{\partial\delta\overline{\psi}}{\partial\overline{\psi}}\gamma_\mu\psi\right), \tag{29b}$$

$$\overline{B}_\mu = \frac{\partial\delta x_\mu}{\partial\psi} + \frac{\mathrm{i}}{2}\left(\delta\overline{\psi}\gamma_\mu + \overline{\psi}\gamma_\mu\frac{\partial\delta\psi}{\partial\psi} + \frac{\partial\delta\overline{\psi}}{\partial\psi}\gamma_\mu\psi\right) \tag{29c}$$

(here the lowering and raising index operations with the Greek indices $\mu, \nu \ldots$ are defined with the metric $\eta_{\mu\nu}$).

Using the definition (16) of the supervelocities $v^\mu$ we can write the final expression for $\delta L$

$$\delta L = \left[\partial x^\mu + \frac{\mathrm{i}}{2}\left(\overline{\psi}\gamma^\mu\partial\psi - \partial\overline{\psi}\gamma^\mu\psi\right)\right] \times \left[A_{\mu\nu}\partial x^\nu + \partial\overline{\psi}B_\mu + \overline{B}_\mu\partial\psi\right],$$

and, after comparing the factors corresponding to the different products of $\partial z^A$, we obtain the following Killing equations:

$$A_{\mu\nu} + A_{\nu\mu} = 0, \tag{30a}$$

$$\overline{B}_\nu + \frac{\mathrm{i}}{2}\overline{\psi}\gamma^\mu A_{\mu\nu} = 0, \quad B_\nu - \frac{\mathrm{i}}{2}\gamma^\mu\psi A_{\mu\nu} = 0, \tag{30b}$$

$$(B_\mu)_b(\overline{\psi}\gamma^\mu)^a - (\gamma^\mu\psi)_b(\overline{B})^a = 0, \tag{30c}$$

$$(\overline{\psi}\gamma^\mu)_a(\overline{B}_\mu)_b - (\overline{\psi}\gamma^\mu)_b(\overline{B}_\mu)_a = 0, \tag{30d}$$

$$(\gamma^\mu\psi)^a(B_\mu)^b - (\gamma^\mu\psi)^b(B_\mu)^a = 0. \tag{30e}$$

The Killing equations (30) are partial differential equations of the first order. In the case of the variations determined by the functional structure $\delta\psi = \delta\psi(\psi)$, $\delta\overline{\psi} = \delta\overline{\psi}(\overline{\psi})$, these equations become simpler ones and the objects (29) are reduced to

$$A_{\mu\nu} = \frac{\partial\delta x_\mu}{\partial x^\nu}, \quad B^\mu = \frac{\partial\delta x^\mu}{\partial\overline{\psi}} - \frac{\mathrm{i}}{2}\left(\gamma^\mu\delta\psi + \frac{\partial\delta\overline{\psi}}{\partial\overline{\psi}}\gamma^\mu\psi\right),$$

$$\overline{B}^\mu = \frac{\partial\delta x^\mu}{\partial\psi} + \frac{\mathrm{i}}{2}\left(\delta\overline{\psi}\gamma^\mu + \overline{\psi}\gamma^\mu\frac{\partial\delta\psi}{\partial\psi}\right).$$

The Killing equations for the spatial coordinates $x^\mu$ read

$$\frac{\partial\delta x_\mu}{\partial x^\nu} + \frac{\partial\delta x_\nu}{\partial x^\mu} = 0.$$

Its solutions independent on $\psi$ and $\overline{\psi}$ are well known; they describe the representation of the Poincaré group $\mathsf{ISO}(p, q)$, $p + q = n$:

$$\delta x^\mu = \varepsilon^\mu + \varepsilon^\mu{}_\nu x^\nu = \varepsilon^\mu + \frac{1}{2}\varepsilon^{\alpha\beta}(\delta^\mu_\alpha\eta_{\nu\beta} - \delta^\mu_\beta\eta_{\nu\alpha})x^\nu. \tag{31}$$

Here $\varepsilon^\mu$ are the infinitesimal parameters of the translation subgroup $\mathsf{T}_n$ and the infinitesimal parameters $\varepsilon^{\mu\nu} = -\varepsilon^{\nu\mu}$ are related to the Lorentz transformation group $\mathsf{SO}(p, q)$. It is easy to find the solution for the variations of the spinor coordinates $\psi$ and $\overline{\psi}$,

$$\delta\psi = -\frac{\mathrm{i}}{2}\hat{\varepsilon}\psi, \quad \delta\overline{\psi} = \frac{\mathrm{i}}{2}\overline{\psi}\hat{\varepsilon}, \tag{32}$$

where

$$\hat{\varepsilon} = \varepsilon_{\mu\nu}\sigma^{\mu\nu}, \quad \sigma^{\mu\nu} = \frac{\mathrm{i}}{4}[\gamma^\mu, \gamma^\nu].$$

In the case of these variations the functions (29) are reduced to

$$A_{\mu\nu} = \varepsilon_{\mu\nu}, \quad B^\mu = \frac{1}{4}[\gamma^\mu, \hat{\varepsilon}]\psi, \quad \overline{B}^\mu = -\frac{1}{4}\overline{\psi}[\gamma^\mu, \hat{\varepsilon}].$$

Taking into account the property of the Dirac matrices $[\sigma^{\alpha\beta}, \gamma^\mu] = \mathrm{i}(\gamma^\alpha\eta^{\mu\beta} - \gamma^\beta\eta^{\mu\alpha})$, one can observe that

$$[\hat{\varepsilon}, \gamma^\mu] = -2\mathrm{i}\varepsilon^\mu{}_\alpha\gamma^\alpha;$$

then

$$\delta v^\mu = \varepsilon^\mu{}_\alpha v^\alpha.$$

Hence, using (26), one can draw the conclusion that $\delta L = 0$ for the transformation with the parameters $\varepsilon^\mu$ and $\varepsilon^{\mu\nu}$.

On the other hand, under the same conditions there exists a solution of the Killing equations with the functional structure $\delta x^\mu = \delta x^\mu(\psi, \overline{\psi})$, which has the following form:

$$\delta x^\mu = \frac{\mathrm{i}}{2}\left(\overline{\psi}\gamma^\mu\xi - \overline{\xi}\gamma^\mu\psi\right), \quad \delta\psi = \xi, \quad \delta\overline{\psi} = \overline{\xi},$$

where $\xi$, $\overline{\xi}$ are infinitesimal spinor parameters. For these transformations we have $B^\mu = 0$, $\overline{B}^\mu = 0$, $A_{\mu\nu} = 0$; therefore $\delta v^\mu = 0$ and $\delta L \equiv 0$. Such a transformation describes the supertranslations with the parameters $\xi$ and $\overline{\xi}$.

It should be noted here that in principle the Killing equations may possess a wider class of solutions than described above, but in our consideration this is not essential.

## 5 Lagrangian in quantum theory

In quantum theory the coordinates $\{x^\mu, \psi, \overline{\psi}\}$ become operators that are, generally speaking, non-commutative objects. Before writing down the Lagrangian and other standard constructions explicitly the manner of operator ordering in products has to be decided on. Let us introduce the generalization of a Jordan product for the case of the presence of odd variables

$$a \circ b := \frac{1}{2}(ab + s(a,b)ba), \tag{33}$$

where $s(a,b)$ is the signature factor for the operators $a$ and $b$ defined in (1).

The velocity operator can be defined by

$$v^\mu = \partial x^\mu + \frac{\mathrm{i}}{2}\left(\overline{\psi}\gamma^\mu \circ \partial\psi - \partial\overline{\psi}\gamma^\mu \circ \psi\right). \tag{34}$$

As to the operator properties of dynamical variables we assume that the momentum operator $p_\mu$ commutes with $\psi$ and $\overline{\psi}$ (this assumption will be confirmed below). Then the quantum Lagrangian receives the form

$$L = \frac{1}{2} v^\mu \eta_{\mu\nu} v^\nu. \tag{35}$$

This expression is invariant under the Poincaré group transformations.

The variation of (35) under an arbitrary synchronous coordinate transformation $z^A \to z'^a = z^A + \delta z^A$ reads

$$\begin{aligned} \delta L = p_\mu \circ \delta v^\mu = \partial \mathcal{Q} \\ + \frac{\mathrm{i}}{2}\left[\delta\overline{\psi} \circ (\hat{p}\partial\psi + \partial(\hat{p}\psi)) - \left(\partial\overline{\psi}\hat{p} + \partial(\overline{\psi}\hat{p})\right)\right] \\ - \partial p_\mu \circ \delta x^\mu, \end{aligned} \tag{36}$$

where

$$\mathcal{Q} = p_\mu \circ \delta x^\mu + \frac{\mathrm{i}}{2}\left(\overline{\psi}\hat{p} \circ \delta\psi - \delta\overline{\psi}\hat{p} \circ \psi\right), \quad \hat{p} = p_\mu \gamma^\mu. \tag{37}$$

Here we have used the assumptions $[p_\mu, \psi] = 0$, $[p_\mu, \overline{\psi}] = 0$, so that (36) holds for any operator properties of $\delta z$.

When the variation "$\delta$" is related to permissible variations, i.e. $\delta L = 0$ (because $L$ is a purely kinematic object), the dynamical equations obtain the form

$$\partial p_\mu = 0, \quad \hat{p}\partial\psi = 0, \quad \partial\overline{\psi}\hat{p} = 0. \tag{38}$$

In addition, in the case of permissible variations there is the following conservation law:

$$\partial \mathcal{Q} = 0.$$

This means that $\mathcal{Q}$ is the generator of permissible variations.

In quantum theory the permissible variations determined by $\delta L = 0$ without any referring to the equations of motion are described by the properties that are different from the language of the classical Killing equations (we cannot give the rigorous definition of the derivatives with respect to non-commutative operators $x^\mu$, $\psi$, $\overline{\psi}$). Nevertheless, a direct calculation shows that supertranslations and $\mathsf{SO}(p,q)$-transformations are permissible variations. Let us find the explicit expressions for the generators of these transformations.

In the case of the supertranslations one can write

$$\delta x^\mu = \varepsilon^\mu + \frac{\mathrm{i}}{2}\left(\overline{\psi}\gamma^\mu\xi - \overline{\xi}\gamma^\mu\psi\right), \quad \delta\psi = \xi, \quad \delta\overline{\psi} = \overline{\xi}, \tag{39}$$

and the generator reads

$$G = \varepsilon^\mu p_\mu + \mathrm{i}(\overline{\psi}\hat{p}\xi - \overline{\xi}\hat{p}\psi) \tag{40}$$

(as follows from (37)).

Similarly, taking the variation related to $\mathsf{SO}(p,q)$-transformations one obtains

$$\delta x^\mu = \varepsilon^\mu{}_\nu x^\nu, \quad \delta\psi = -\frac{\mathrm{i}}{2}\hat{\varepsilon}\psi, \quad \delta\overline{\psi} = \frac{\mathrm{i}}{2}\overline{\psi}\hat{\varepsilon}, \tag{41}$$

with

$$G = -\frac{1}{2}\varepsilon^{\mu\nu} J_{\mu\nu}, \quad J_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}, \tag{42}$$

where

$$L_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu, \quad S_{\mu\nu} = -\frac{1}{2}\overline{\psi}\{\hat{p}, \sigma_{\mu\nu}\}\psi. \tag{43}$$

Here we assume that $[x_\mu, p_\nu] = [x_\nu, p_\mu]$; therefore, the symmetrization in the expression of $L_{\mu\nu}$ is unnecessary. The curly brackets denote the anticommutator of *matrices*. In our context the operators $L_{\mu\nu}$ and $S_{\mu\nu}$ have the meaning of orbital and spin momenta of a superparticle.

Basing ourselves on (40) and (42) it is possible to develop the algebra of the commutation relations; these analyses will be carried out in the next section.

## 6 Commutation relations and charges

If $G$ denotes the generator of permissible variations then the variation of an arbitrary operator $\mathcal{F}$ obeys the equation

$$\delta\mathcal{F} = \frac{1}{\mathrm{i}\hbar}[\mathcal{F}, G]. \tag{44}$$

When $\mathcal{F}$ and $G$ are given and the algebra of the commutation relations is known, the variation $\delta\mathcal{F}$ can be directly calculated from (44). In our context we have the explicit forms for $\mathcal{F}$ and $G$ (determined from the analyses of the symmetries of $L$), and (44) allows one to obtain some information about the operator algebra using the properties (given a priori) of the system under consideration.

Making the substitutions of the different operators $\mathcal{F}$ with the same generator $G$ one arrives at the algebraic system for the unknown commutators which is totally or partially solvable depending on the character of the model.

In order to construct the algebra of the commutation relations let us consider supertranslations as the permissible variations. The corresponding generator can be represented by

$$G = p_\mu \varepsilon^\mu + \overline{Q}\xi + \bar{\xi}Q, \tag{45}$$

where

$$\overline{Q} = \mathrm{i}\overline{\psi}\hat{p}, \quad Q = -\mathrm{i}\hat{p}\psi.$$

Using the definition of the variations (39) we can obtain after a simple calculation the following commutation relations for the spinor coordinates:

$$\begin{aligned} &[\psi, p_\mu] = 0, \quad [\psi, \overline{\psi}\hat{p}] = \hbar\mathbb{I}, \quad [\psi, \hat{p}\psi] = 0, \\ &[\overline{\psi}, p_\mu] = 0, \quad [\overline{\psi}, \hat{p}\psi] = \hbar\mathbb{I}, \quad [\overline{\psi}, \overline{\psi}\hat{p}] = 0, \end{aligned} \tag{46}$$

where $\mathbb{I}$ denotes the unit matrix with spinor indices. Analogously, for the coordinates of $\mathbb{M}^n$ we can write

$$[x^\mu, p_\nu] = \mathrm{i}\hbar\delta^\mu_\nu, \quad [x^\mu, \overline{\psi}\hat{p}] = \frac{\mathrm{i}\hbar}{2}\overline{\psi}\gamma^\mu, \quad [x^\mu, \hat{p}\psi] = \frac{\mathrm{i}\hbar}{2}\gamma^\mu\psi. \tag{47}$$

Note that (46) is consistent with the initial assumption about the commutativity of the momentum $p_\mu$ with the operators $\psi$ and $\overline{\psi}$.

Putting $p_\mu$ instead of $\mathcal{F}$ in the formula (44) we obtain

$$[p_\mu, p_\nu] = 0, \quad [p_\mu, \overline{\psi}\hat{p}] = 0, \quad [p_\mu, \hat{p}\psi] = 0. \tag{48}$$

Taking into account the commutativity of the momenta $p_\mu$, we can define the operator $\hat{\alpha}$ inverse to $\hat{p}$ by

$$\hat{p}\hat{\alpha} = \hat{\alpha}\hat{p} = \mathbb{I},$$

where the property of commutativity is related to the matrix elements[1]. It is possible to give an explicit expression for $\hat{\alpha}$. Evidently, $\hat{p}\hat{p} = p_\mu p_\nu \gamma^\mu\gamma^\nu = \eta^{\mu\nu}p_\mu p_\nu$, so that

$$\hat{\alpha} = \frac{\hat{p}}{\eta^{\mu\nu}p_\mu p_\nu} := \alpha_\mu\gamma^\mu, \quad \alpha_\mu = \frac{p_\mu}{\eta^{\mu\nu}p_\mu p_\nu}.$$

Then, using the formulae (46)–(48) we can write

$$[\psi, \psi] = 0, \quad [\overline{\psi}, \overline{\psi}] = 0, \quad [\psi, \overline{\psi}] = \hbar\mathbb{I}, \tag{49a}$$

$$[\psi, p_\mu] = 0, \quad [\overline{\psi}, p_\mu] = 0, \tag{49b}$$

$$[x^\mu, p_\nu] = \mathrm{i}\hbar\delta^\mu_\nu, \quad [p_\mu, p_\nu] = 0, \tag{49c}$$

$$[x^\mu, \psi] = -\frac{\mathrm{i}\hbar}{2}\hat{\alpha}\gamma^\mu\psi, \quad [x^\mu, \overline{\psi}] = -\frac{\mathrm{i}\hbar}{2}\overline{\psi}\gamma^\mu\hat{\alpha}. \tag{49d}$$

[1] That is, $(\hat{\alpha})^a_b(\hat{p})^b_c = (\hat{p})^b_c(\hat{\alpha})^a_b = \delta^a_c$.

It is useful to write down the commutators between $x^\mu$ and the operators $\hat{p}$ and $\hat{\alpha}$, which can be calculated immediately from the definitions of these objects:

$$[x^\mu, \hat{p}] = \mathrm{i}\hbar\gamma^\mu, \quad [x^\mu, \hat{\alpha}] = -\mathrm{i}\hbar\hat{\alpha}\gamma^\mu\hat{\alpha}. \tag{50}$$

The commutator $[x^\mu, x^\nu]$ can be obtained in an indirect way with the use of the Jacobi identity for the triple of operators $x^\mu$, $x^\nu$ and $\psi$. It is easy to prove that

$$[x^\mu, x^\nu] = \frac{\hbar}{4}\overline{\psi}(\gamma^\mu\hat{\alpha}\gamma^\nu - \gamma^\nu\hat{\alpha}\gamma^\mu) \circ \psi. \tag{51}$$

Using the properties of the $\gamma$-matrices one can rewrite (51) as

$$[x^\mu, x^\nu] = \frac{\mathrm{i}\hbar}{2}\overline{\psi}\{\hat{\alpha}, \sigma^{\mu\nu}\} \circ \psi, \tag{52}$$

where the curly brackets are related to the anticommutator of matrices.

The commutation relations (52) show that in the quantum theory of a superparticle the coordinates $x^\mu$, $\psi$ and $\overline{\psi}$ are not commutative. Therefore the features of quantum theory on a superspace essentially differ from the corresponding theory on a usual Riemannian manifold. The explicit manifestation of them consists in the noncommutativity of the geometrical coordinates $\{x^\mu\}$ related to the underlying manifold $\mathbb{M}^n$.

Now, using the commutative relations obtained above, let us consider the algebra of charges describing the $\mathsf{SO}(p,q)$ symmetry. We have

$$\delta x^\mu = \varepsilon^\mu{}_\nu x^\nu, \quad \delta\psi = -\frac{\mathrm{i}}{2}\hat{\varepsilon}\psi, \quad \delta\overline{\psi} = \frac{\mathrm{i}}{2}\overline{\psi}\hat{\varepsilon}.$$

The generator of these transformations has the form

$$G = -\frac{1}{2}\varepsilon^{\mu\nu}J_{\mu\nu}, \tag{53}$$

where

$$\begin{aligned} J_{\mu\nu} &= L_{\mu\nu} + S_{\mu\nu}, \quad L_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu, \\ S_{\mu\nu} &= -\frac{1}{2}\overline{\psi}\{\hat{p}, \sigma_{\mu\nu}\} \circ \psi. \end{aligned}$$

Then, making the substitution of the variations of charges describing supertranslations and $\mathsf{SO}(p,q)$-rotations to the main variational equation (44) with a total generator including these transformations, one gets the following algebra of the charges:

$$[p_\mu, p_\nu] = 0, \quad [p_\mu, Q] = 0, \quad [p_\mu, \overline{Q}] = 0, \tag{54a}$$

$$[p_\mu, J_{\alpha\beta}] = \mathrm{i}\hbar(p_\alpha\eta_{\mu\beta} - p_\beta\eta_{\mu\alpha}), \tag{54b}$$

$$[Q, \overline{Q}] = \hbar\hat{p}, \quad [\overline{Q}, J_{\mu\nu}] = \mathrm{i}\hbar\overline{Q}\sigma_{\mu\nu}, \tag{54c}$$

$$[Q, J_{\mu\nu}] = -\mathrm{i}\hbar\sigma_{\mu\nu}Q, \tag{54d}$$

$$[J_{\mu\nu}, J_{\alpha\beta}] = \mathrm{i}\hbar(\eta_{\mu\beta}J_{\nu\alpha} + \eta_{\mu\alpha}J_{\mu\beta} - \eta_{\nu\beta}J_{\mu\alpha} - \eta_{\mu\alpha}J_{\nu\beta}). \tag{54e}$$

The algebra (54) coincides with the usual algebra of the supersymmetrical extension of the inhomogeneous Lorentz group that has been introduced in [4,5] in somewhat different notations. It should be noted here that because of $[x^\mu,\psi]\neq 0$, $[x^\mu,\overline{\psi}]\neq 0$, $[x^\mu,x^\nu]\neq 0$ the operators $L_{\mu\nu}$ and $S_{\mu\nu}$ considered separately do not form the Lie algebra of the form (54) and due to this have no independent physical meaning. As an integral of motion appears their combination $J_{\mu\nu}$.

In order to construct the coordinate representation and the space of states, it is necessary to determine the basis of the Hilbert space of states $\mathcal{H}$. As is well known, this basis is generated by a spectral problem of some total set of commuting observables. In the superspace the coordinates $\{z^A\}$ do not commute with each other and therefore do not form a total set (as it holds in the usual quantum mechanics on the Riemannian manifold). The total set can be constructed as a set of combinations of the existing variables.

Define the operators

$$\eta^\mu_\pm = x^\mu \pm \frac{\mathrm{i}}{2}\overline{\psi}\gamma^\mu\circ\psi. \tag{55}$$

A direct calculation leads to the following commutation relations:

$$[\eta^\mu_+,\psi] = -\mathrm{i}\hbar\hat{\alpha}\gamma^\mu\psi, \quad [\eta^\mu_+,\overline{\psi}] = 0, \tag{56a}$$
$$[\eta^\mu_-,\overline{\psi}] = -\mathrm{i}\hbar\overline{\psi}\gamma^\mu\hat{\alpha}, \quad [\eta^\mu_-,\psi] = 0, \tag{56b}$$
$$[\eta^\mu_\pm,\eta^\nu_\pm] = 0, \quad [\eta^\mu_\pm,\eta^\nu_\mp] = -[\eta^\mu_\mp,\eta^\nu_\pm], \tag{56c}$$
$$[\eta^\mu_\pm,\eta^\nu_\mp] = \pm\frac{\hbar}{2}\overline{\psi}(\gamma^\mu\hat{\alpha}\gamma^\nu-\gamma^\nu\hat{\alpha}\gamma^\mu)\circ\psi, \tag{56d}$$

and

$$[\eta^\mu_\pm,p_\nu] = \mathrm{i}\hbar\delta^\mu_\nu, \tag{57}$$
$$[\eta^\mu_+,x^\nu] = -\frac{\hbar}{2}\gamma^\nu\hat{\alpha}\gamma^\mu\circ\psi,$$
$$[\eta^\mu_-,x^\nu] = \frac{\hbar}{2}\gamma^\mu\hat{\alpha}\gamma^\nu\circ\psi. \tag{58}$$

These results can be obtained immediately from the main variational equation with the generator of supertranslations, where one can put

$$\delta\eta^\mu_+ = \varepsilon^\mu+\overline{\psi}\gamma^\mu\xi, \quad \delta\eta^\mu_- = \varepsilon^\mu-\overline{\xi}\gamma^\mu\psi.$$

The commutators between $\eta^\mu_\pm$ and $Q$, $\overline{Q}$ are

$$[\eta^\mu_+,Q] = 0, \quad [\eta^\mu_+,\overline{Q}] = \mathrm{i}\hbar\overline{\psi}\gamma^\mu,$$
$$[\eta^\mu_-,Q] = \mathrm{i}\hbar\gamma^\mu\psi, \quad [\eta^\mu_-,\overline{Q}] = 0.$$

One can observe that basing oneself on the system of operators $(x^\mu,\psi,\overline{\psi})$ it is possible to construct two separated sets of commuting observables $(\eta^\mu_+,\overline{\psi})$ and $(\eta^\mu_-,\psi)$, which are not commutative with each other.

Note that the velocity operator can be rewritten as

$$v^\mu = \partial\eta^\mu_+ - \mathrm{i}\partial\overline{\psi}\gamma^\mu\circ\psi = \partial\eta^\mu_- + \mathrm{i}\overline{\psi}\gamma^\mu\circ\partial\psi.$$

Evidently, the derivatives of $\psi$ with respect to $\tau$ do not appear in the explicit expression of the Lagrangian represented in terms of $(\eta^\mu_+,\overline{\psi})$; these derivatives are included in the definition of the momenta conjugate to $\overline{\psi}$,

$$L = \frac{1}{2}p_\mu\circ v^\mu = \frac{1}{2}\left[\partial\eta^\mu_+\circ p_\mu + \partial\overline{\psi}\circ(-\mathrm{i}\hat{p}\psi)\right]$$
$$\equiv \frac{1}{2}\left(\partial\eta^\mu_+\circ p_\mu + \partial\overline{\psi}\circ Q\right).$$

It is easy to observe that in such a definition the variables $\psi$ have no corresponding conjugate momenta (expressed as derivatives of $L$ with respect to $\partial\psi$ with a classical meaning). Therefore, the theory possesses constraints.

## 7 Coordinate representation

Let us choose the set of operators $\zeta = (\eta^\mu_+,\overline{\psi})$ as the total set of commutative observables; for the sake of abbreviation we will write $\eta^\mu$ instead of $\eta^\mu_+$. Define the basis states by

$$\zeta|\zeta'\rangle = \zeta'|\zeta'\rangle,$$

or, in more details,

$$\eta^\mu|\zeta'\rangle = \eta'^\mu|\zeta'\rangle, \quad \overline{\psi}|\zeta'\rangle = \overline{\psi}'|\zeta'\rangle. \tag{59}$$

We choose the normalization condition in the standard form

$$\langle\zeta'|\zeta''\rangle = \delta(\zeta'-\zeta''), \tag{60}$$

and for an arbitrary function $f(\zeta')$ we can write

$$\int f(\zeta'')\delta(\zeta'-\zeta'')\mathrm{d}\zeta'' = f(\zeta'),$$

where $\delta(\zeta'-\zeta'') = \delta(\eta'-\eta'')\delta(\overline{\psi}'-\overline{\psi}'')$. In particular, the $\delta$-function for spinor (Grassmannian) variables reads

$$\delta(\overline{\psi}'-\overline{\psi}'') = \overline{\psi}'-\overline{\psi}''$$

and $\delta(0)=1$ (differential and integral calculus for Grassmannian numbers are discussed in [7]).

Now let us consider the matrix elements of the operators, playing an important role in the construction of the coordinate representation. Evidently, in the case of operators which form the total set we can write

$$\langle\zeta'|\overline{\psi}|\zeta''\rangle = \overline{\psi}''\langle\zeta'|\zeta''\rangle, \quad \langle\zeta'|\eta^\mu|\zeta''\rangle = \eta''^\mu\langle\zeta'|\zeta''\rangle. \tag{61}$$

The matrix elements for the momentum operators $p_\mu$ can be obtained with the use of

$$[\eta^\mu,p_\nu] = \mathrm{i}\hbar\delta^\mu_\nu, \quad [p_\mu,p_\nu] = 0,$$

so it is easy to find that

$$\langle\zeta'|p_\mu|\zeta''\rangle = -\mathrm{i}\hbar\frac{\partial}{\partial\eta''^\mu}\langle\zeta'|\zeta''\rangle + \frac{\partial F(\zeta'')}{\partial\eta''^\mu}\langle\zeta'|\zeta''\rangle, \tag{62}$$

where $F(\zeta)$ is some continuous function. Analogously, from

$$[\overline{\psi},Q] = \mathrm{i}\hbar\mathbb{I}, \quad [\overline{\psi},\overline{\psi}] = 0, \quad [Q,Q] = 0,$$

we get

$$\langle\zeta'|Q|\zeta''\rangle = -\mathrm{i}\hbar\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}\overline{\psi}''}\langle\zeta'|\zeta''\rangle + \frac{\boldsymbol{\partial}\Omega(\zeta'')}{\boldsymbol{\partial}\overline{\psi}''}, \tag{63}$$

where we have taken into account the fact that $\langle\zeta'|\zeta''\rangle$ is an odd number. As in (62), $\Omega(\zeta'')$ denotes some continuous function. Here and below the derivation with respect to $\overline{\psi}$ is regarded to be a left one.

Since $[p_\mu, Q] = 0$,

$$\frac{\partial}{\partial\eta''^\mu}\frac{\partial}{\partial\overline{\psi}''}\Omega(\zeta'') = \frac{\partial}{\partial\overline{\psi}''}\frac{\partial}{\partial\eta''^\mu}F(\zeta''), \tag{64}$$

and to a certain extent $\Omega$ can be identified with $F$ (they can differ on a linear function of $\zeta$). Moreover, $\Omega$ and $F$ can be removed by a unitary transformation. So we can put $\Omega \equiv 0$, $F \equiv 0$.

Define the wave function for the state $|\Phi\rangle$ as a scalar product

$$\Phi(\zeta) = \langle\zeta|\Phi\rangle. \tag{65}$$

Then the coordinate representation of the operators considered above has the form

$$\begin{aligned}(\overline{\psi}\Phi)(\zeta') &= \overline{\psi}'\Phi(\zeta'),\\ (\eta^\mu\Phi)(\zeta') &= \eta'^\mu\Phi(\zeta'),\\ (p_\mu\Phi)(\zeta') &= -\mathrm{i}\hbar\frac{\partial}{\partial\eta'^\mu}\Phi(\zeta'),\end{aligned} \tag{66a}$$

$$(Q\Phi)(\zeta') = -\mathrm{i}\hbar\frac{\partial}{\partial\overline{\psi}'}\Phi(\zeta'). \tag{66b}$$

Using the results obtained one can easily get the coordinate representation for the generator of supertranslations on wave functions:

$$G = \varepsilon^\mu p_\mu + \overline{\xi}Q + \overline{Q}\xi.$$

Since the variations of $\overline{\psi}$ and $\eta^\mu$ are

$$\delta\overline{\psi} = \overline{\xi}, \quad \delta\eta^\mu = \varepsilon^\mu + \mathrm{i}\overline{\psi}\gamma^\mu\xi,$$

then the change of the wave function under such variations of its arguments reads

$$\begin{aligned}\delta\Phi(\zeta) \equiv \delta\zeta\frac{\partial\Phi}{\partial\zeta} &= \delta\eta^\mu\frac{\partial\Phi}{\partial\eta^\mu} + \delta\overline{\psi}\frac{\partial\Phi}{\partial\overline{\psi}}\\ &= (\varepsilon^\mu + \mathrm{i}\overline{\psi}\gamma^\mu\xi)\frac{\partial\Phi}{\partial\eta^\mu} + \overline{\xi}\frac{\partial\Phi}{\partial\overline{\psi}}.\end{aligned}$$

On the other hand, according to the definition of the generator we conclude that $\delta\Phi = (\mathrm{i}/\hbar)G\Phi$; then taking into account the coordinate representation of the operators $p_\mu$, $Q$ and $\overline{\psi}$, we arrive at the following expression:

$$\begin{aligned}(G\Phi)(\zeta) &= -\mathrm{i}\hbar\varepsilon^\mu\frac{\partial\Phi(\zeta)}{\partial\eta^\mu}\\ &\quad + \mathrm{i}\overline{\psi}\left(-\gamma^\mu\frac{\partial}{\partial\eta^\mu}\right)\xi\Phi(\zeta) - \mathrm{i}\hbar\overline{\xi}\frac{\partial}{\partial\overline{\psi}}\Phi(\zeta)\\ &= -\mathrm{i}\hbar\left[(\varepsilon^\mu + \overline{\psi}\gamma^\mu\xi)\frac{\partial}{\partial\eta^\mu} + \overline{\xi}\frac{\partial}{\partial\overline{\psi}}\right]\Phi(\zeta),\end{aligned} \tag{67}$$

which is in agreement with the previous conclusion.

Let us find the coordinate representation of the other operators appearing in the commutation relations. In particular, consider the operator $\psi$. Since

$$[\overline{\psi}, p_\mu] = 0, \quad [\psi, p_\mu] = 0,$$

then

$$\frac{\partial}{\partial\eta''^\mu}\langle\zeta'|\psi|\zeta''\rangle = -\frac{\partial}{\partial\eta'^\mu}\langle\zeta'|\psi|\zeta''\rangle \tag{68}$$

(the same equality trivially holds also for $\overline{\psi}$ due to $\langle\zeta'|\overline{\psi}|\zeta''\rangle = \overline{\psi}''\langle\zeta'|\zeta''\rangle$). In a similar way, from $[\psi, Q] = 0$ we get

$$\frac{\partial}{\partial\overline{\psi}'}\langle\zeta'|\psi|\zeta''\rangle = \frac{\partial}{\partial\overline{\psi}''}\langle\zeta'|\psi|\zeta''\rangle. \tag{69}$$

Further, using the definition $Q = -\mathrm{i}\hat{p}\psi$ we can write

$$\begin{aligned}&-\mathrm{i}\hbar\frac{\partial}{\partial\overline{\psi}''}\langle\zeta'|\zeta''\rangle\\ &\equiv \langle\zeta'|Q|\zeta''\rangle = -\mathrm{i}\int \mathrm{d}\zeta'''\langle\zeta'|\hat{p}|\zeta'''\rangle\langle\zeta'''|\psi|\zeta''\rangle\\ &= -\hbar\int \mathrm{d}\zeta'''\left(\gamma^\mu\frac{\partial}{\partial\eta'''^\mu}\langle\zeta'|\zeta'''\rangle\right)\langle\zeta'''|\psi|\zeta''\rangle\\ &= \hbar\gamma^\mu\frac{\partial}{\partial\eta'^\mu}\langle\zeta'|\psi|\zeta''\rangle,\end{aligned}$$

and, taking into account (68) we conclude that the matrix element of the operator $\psi$ is the solution of the differential equation

$$\mathrm{i}\frac{\partial}{\partial\overline{\psi}''}\langle\zeta'|\zeta''\rangle = \gamma^\mu\frac{\partial}{\partial\eta''^\mu}\langle\zeta'|\psi|\zeta''\rangle. \tag{70}$$

This equality means that the operator $\psi$ is represented on wave functions by an integral operator. Multiplying (70) by the wave function and taking the integral of $\zeta$ (or $\zeta''$) we find that

$$\frac{\partial}{\partial\overline{\psi}'}\Phi(\zeta') + \mathrm{i}\gamma^\mu\psi\frac{\partial}{\partial\eta'^\mu}\Phi(\zeta') = 0 \tag{71}$$

(here the action of the operator $\psi$ precedes the differentiation).

Equation (71) can be chosen as the definition of the realization of the operator $\psi$; it can be rewritten

$$(\psi\Phi)(\zeta) = -\mathrm{i}[\hat{\partial}]^{-1}\left(\frac{\partial}{\partial\overline{\psi}}\Phi(\zeta)\right), \tag{72}$$

where $[\hat{\partial}]^{-1}$ denotes the operator inverse to $\hat{\partial} = \gamma^\mu\partial/\partial\eta^\mu$.

The action of the operator $\overline{Q} = \mathrm{i}\overline{\psi}\hat{p}$ on wave functions is described by

$$(\overline{Q}\Phi)(\zeta') = -\left(\hbar\overline{\psi}'\gamma^\mu\frac{\partial}{\partial\eta'^\mu}\right)\Phi(\zeta'). \tag{73}$$

To analyze the properties of the matrix elements of $\hat{\alpha}$ we take the matrix element of the equality $\hat{p}\hat{\alpha} = \hat{\alpha}\hat{p} = \mathbb{I}$; using the representation of $p_\mu$ we get

$$\langle\zeta'|\hat{p}\circ\hat{\alpha}|\zeta''\rangle = -\frac{\mathrm{i}\hbar}{2}(\hat{\partial}'' - \hat{\partial}')\langle\zeta'|\hat{\alpha}|\zeta''\rangle \equiv \langle\zeta'|\zeta''\rangle,$$

where the matrix indices of $\hat{\alpha}$ are contracted with the matrix indices of $\gamma^\mu$. The last equality shows that $\hat{\alpha}$ is realized by the integral operator inverse to $\hat{\partial}$

$$\hat{p}\Phi = -\mathrm{i}\hbar\frac{\partial}{\partial\eta^\mu}\Phi, \quad \hat{\alpha}\left(\mathrm{i}\hbar\gamma^\mu\frac{\partial}{\partial\eta^\mu}\Phi\right) = -\Phi.$$

Since $[\overline{\psi}, \psi] = \hbar\hat{\alpha}$, it is easy to obtain the equation connecting the matrix elements of $\psi$ and $\hat{\alpha}$:

$$\hbar\langle\zeta'|\hat{\alpha}|\zeta''\rangle = (\overline{\psi}' - \overline{\psi}'')\langle\zeta'|\psi|\zeta''\rangle.$$

Besides $p_\mu$, $Q$, $\overline{Q}$, the theory possesses another conserved charge, namely the energy, which corresponds to time shifts and is described by the Hamiltonian operator

$$H = L = \frac{1}{2}\eta^{\mu\nu}p_\mu p_\nu. \tag{74}$$

The conservation law $H = E = \text{const}$ implies the following condition on the stationary states:

$$-\frac{\hbar^2}{2}\eta^{\mu\nu}\frac{\partial}{\partial\eta^\mu}\frac{\partial}{\partial\eta^\nu}\Phi = E\Phi, \tag{75}$$

which corresponds to the Schrödinger equation.

Thus, the realization space for the operators of the theory under consideration is constructed in outline. The action of the geometric coordinate operators $x^\mu$ on wave functions can be obtained from the definition $x^\mu = \eta^\mu - (\mathrm{i}/2)\overline{\psi}\gamma^\mu \circ \psi$; the last term causes the integral operator form of $x^\mu$. The realization of the momentum $p_\mu$, conjugate to $x^\mu$ is a differential operator with respect to the artificially introduced variable $\eta^\mu$. Because of $[x^\mu, x^\nu] \neq 0$, now there is not a simple connection between $x^\mu$ and $p_\mu$ like $p_\mu = -\mathrm{i}\hbar\partial/\partial x^\mu$ (moreover, in quantum theory the derivative with respect to $x^\mu$ is not at all defined).

Wave functions depend on the variables $\eta^\mu$ and $\psi$, and the operators $p_\mu$ and $Q$ are described by the derivations with respect to $\eta^\mu$ and $\psi$ (see (66)). Note that in classical theory the derivative

$$\left.\frac{\partial}{\partial\overline{\psi}}\right|_\eta$$

does not coincide with the invariant operator $D$. From

$$\left.\frac{\partial}{\partial\overline{\psi}}\right|_x \Phi(\eta(x,\psi,\overline{\psi}),\overline{\psi}) = \left.\frac{\partial}{\partial\overline{\psi}}\right|_\eta \Phi + \frac{\mathrm{i}}{2}\gamma^\mu\psi\left.\frac{\partial}{\partial\eta^\mu}\right|_{\overline{\psi}}\Phi$$
$$= \left.\frac{\partial}{\partial\overline{\psi}}\right|_\eta \Phi + \frac{\mathrm{i}}{2}\gamma^\mu\psi\left.\frac{\partial}{\partial x^\mu}\right|_{\overline{\psi}}\Phi,$$

we get

$$\left.\frac{\partial}{\partial\overline{\psi}}\right|_\eta = \left(\left.\frac{\partial}{\partial\overline{\psi}}\right|_x - \frac{\mathrm{i}}{2}\gamma^\mu\psi\left.\frac{\partial}{\partial x^\mu}\right|_{\overline{\psi}}\right)\Phi,$$

and this combination differs from $D\Phi$ by a sign. Such an expression is precisely a definition of the action of $Q$ on a superspace (see [5,6]).

Finally, it is worthwhile to remark that, introducing the basis of the Hilbert space of states, we use the new variables $\eta^\mu$ and $\overline{\psi}$. The initial classical Lagrangian can be written in terms of them as

$$L = \frac{1}{2}\eta^{\mu\nu}p_\mu p_\nu = \frac{1}{2}\eta_{\mu\nu}v^\mu v^\nu, \quad v^\mu = \partial\eta^\mu - \mathrm{i}\partial\overline{\psi}\gamma^\mu\psi.$$

The corresponding momenta are

$$\frac{\partial L}{\partial\partial\eta^\mu} = p_\mu = \eta_{\mu\nu}v^\nu, \quad \frac{\partial L}{\partial\partial\overline{\psi}} = Q = -\mathrm{i}\hat{p}\psi, \quad \frac{\partial L}{\partial\partial\psi} = 0,$$

i.e. the theory is constrained. In the present consideration we have not introduced the phase space variables, therefore the constraint on the momenta $Q + \mathrm{i}\hat{p}\psi = 0$ transforms in a natural way to the definition of the coordinate representation of the operator $\psi$.

In a similar manner one can construct the coordinate representation for the total set $(\psi, \eta^\mu_-)$. The basis, generated by these operators, is connected with the one described above by means of a unitary transformation, analogously to the connection between the coordinate and momentum representations in the usual quantum theory.

## 8 Discussion

In the preceding papers [1–3] and in the present one we have considered the extension of Schwinger's quantization procedure to the case of manifolds with a group structure including a superspace. The approach presented may be viewed as an effective method for the study of some quantum models where the physical meaning of a theory is closely connected with its geometric structure. Usually in these cases the canonical quantization postulates are not self-consistent and its use for constructing a theory requires some truthful but not sufficiently argued assumptions. At the same time the extended Schwinger quantization procedure based on symmetry properties of the theory expressed in terms of the framework of Lie groups and algebras allows one to determine the following:

(1) the quantum Lagrangian for a particle on the manifold with a group structure;
(2) the quantum equations of motion for a particle and the corresponding conservation laws;
(3) the algebra of the commutation relations for operators describing a particle;
(4) the coordinate representation of quantum mechanics, i.e. the form of the action of operators on wave functions, the wave equation, etc.

In contradiction to the canonical quantization approach, Schwinger's method in principle does not require the use of phase space variables. Due to this it may happen that in degenerate theories the usual analyses of constraints [8] become unnecessary.

The results obtained in [1–3] and in the present paper satisfy the correspondence principle and are connected in general with the main results of the works of [9–13], where the structure of a theory has been obtained by the use of a canonical quantization procedure supplemented by additional special assumptions.

It seems that the conclusions derived from our series of works may be useful in investigations of the quantum theory of a particle interacting with a gravitational field and in solitonic models of elementary particles in the framework of the collective-coordinate formalism.

## References


1. N.M. Chepilko, A.V. Romanenko, Eur. Phys. J. C **21**, 369 (2001)
2. N.M. Chepilko, A.V. Romanenko, Eur. Phys. J. C **21**, 587 (2001)
3. N.M. Chepilko, A.V. Romanenko, Eur. Phys. J. C **21**, 757 (2001)
4. D.V. Volkov, V.P. Akulov, Phys. Lett. B **46**, 109 (1973)
5. J. Wess, J. Bagger, Supersymmetry and supergravity (Princeton University Press, Princeton, N.Y. 1983)
6. Proceedings of the 1st School on Supergravity, edited by S. Ferrara, J. Tailor (Cambridge University Press, Cambridge 1982)
7. F. Berezin, Introduction to algebra and analysis with anticommuting variables (in Russian) (Moskow University Press, Moskow 1982)
8. P.A.M. Dirac, Lectures on quantum mechanics (Yeshiva Univercity, Academic Press, New York 1967)
9. R. Casalbuoni, Nuovo Cim. A **33**, 389 (1976)
10. A. Barducci et al., Nuovo Cim. A **35**, 377 (1976)
11. V. De Alfaro et al., Phys. Lett. B **200**, 323 (1988)
12. M. Henneaux, C. Teitelboim, Ann. of Phys. **143**, 127 (1982)
13. S.J. Rabello, A.V. Vaidia, Phys. Lett. B **337**, 110 (1994)